Can a String's Tension Exert a Torque on a Pulley?

Dennis E. Krause and Yifei Sun, Wabash College, Crawfordsville, IN

A typical textbook problem in rotational dynamics involves calculating the angular acceleration of a massive pulley due to a string, such as in the example shown in Fig. 1. The string is assumed to be massless and to move without slipping over the pulley, which is mounted on a frictionless axle. If T_L and T_R are the tensions pulling at the left and right edges of the pulley (see Fig. 1), respectively, the net torque on the pulley is then $\tau_{net} = (T_L - T_R)R$, where R is the radius of the pulley. (It is assumed that positive torque corresponds to the counterclockwise direction.) While this analysis, which is typical of what is found in many introductory physics texts,[1] is correct, it should raise several questions in the mind of a student. First, since most texts argue that the tension everywhere in a massless string is constant,[2] why is $T_L \neq T_R$? Second, since tension is an internal force (except at the ends of the string, which are obviously not tied to the pulley),[3] how can tension exert a force and torque on a pulley? In this paper, we will address these questions, which are overlooked in most textbook treatments of this problem whose approach appears inconsistent with the concepts presented elsewhere in the text.

As should be expected, the answers to the above questions can be found in the literature, but surprisingly, they come mainly from the analysis of a related but different problem—the problem of the capstan. Quite some time ago in this journal, Hazelton described how a capstan uses friction as a force multiplier to secure a boat.[4] The treatment is well known for this problem,[5] and various lab exercises have verified its predictions,[6-8] but it is surprising that the same analysis is not also applied to the pulley problem, which is more frequently encountered.

We begin by addressing the first question above on why the tension is not constant in the pulley problem. Let us consider an infinitesimal segment of the string subtending by an angle $d\theta$ as shown in Fig. 2. If the string is massless (so gravity may be neglected), there are four forces acting on this segment. First, there are the two tensions T and $T+dT$ pulling on the segment at angles $d\theta/2$ relative to the tangent line of the pulley at that point. Then the pulley exerts a normal force $n(\theta)d\theta$, directed radially outward on the segment, and a frictional force df. (Here we assume the string would slide counterclockwise around the pulley if there was no friction between the string and pulley, so the direction of df is chosen to counter this motion.) If the string is slipping with respect to the pulley, df would be due to kinetic friction (rather than static friction). If we apply Newton's second law to the tangential components of the forces, we have

$$a_{\tan} dm = [(T+dT)-T]\cos\left(\frac{d\theta}{2}\right) - df, \tag{1}$$

where dm is the mass of the segment. Since $d\theta$ is infinitesimal, $\cos(d\theta/2) = 1$. Finally, since we assume the string is massless, $dm = 0$, which then tells us that the difference in tensions $dT = df$. We can see from this equality that the tension varies in the portion of the string in contact with the pulley due to the existence of friction. Therefore, if there is no friction between the pulley and string, the tension within a massless string is indeed a constant throughout the string, even in this problem. However, in the presence of friction, the tension cannot be the same everywhere, even for a massless string. It is interesting to note that in most such pulley problems, the difference in tensions T_L and T_R is essentially assumed as a fact, which must be true since the pulley experiences an angu-

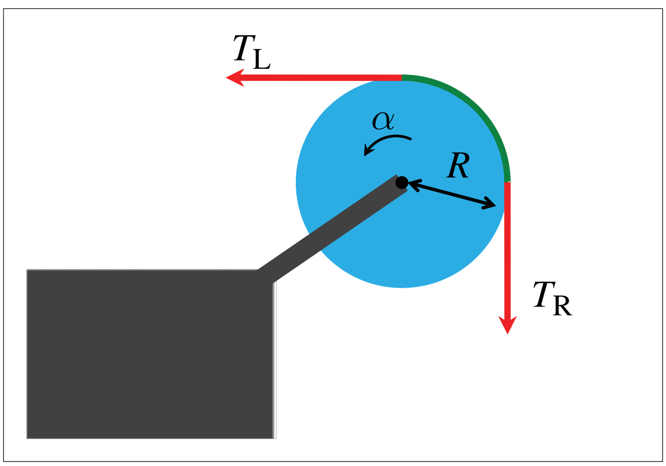

Fig. 1. A free-body diagram that is typical of what is found in textbooks to calculate the torque exerted on a pulley by a string. Here we will assume $T_L > T_R$ so that the pulley's angular acceleration is counterclockwise.

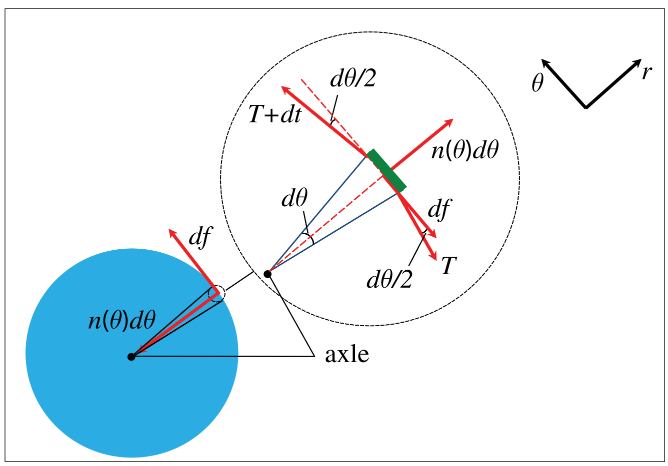

Fig. 2. Here we examine the forces on an infinitesimal segment of the string in contact with the pulley, and the reaction forces that act back on the pulley.

 DOI: 10.1119/1.3566035

lar acceleration, but the origin of this difference is not usually explained.

To solve the second question, how the tension can exert a torque on the pulley, we need only look at the forces the string segment exerts on the pulley, as shown in Fig. 2. There we see that the *tension does not act on the pulley.* Rather, the string segment exerts a frictional force df and a normal force $n(\theta)d\theta$ on the pulley, which are the reaction pairs of the forces of the pulley on the string segment. Furthermore, since the line of action of the normal force acts through the axle, it cannot exert a torque on the pulley; only the frictional force can produce a torque on the pulley.

Let us now calculate the total torque on the pulley due to the frictional force of the entire string. The torque due to the single segment shown in Fig. 2 is $d\tau = Rdf = RdT$, since we have shown that $df = dT$. Since the direction of the torques of each segment is the same, the total torque is then obtained by integrating each part, giving

$$\tau_{\mathrm{net}} = \int_{\tau_{\mathrm{R}}}^{\tau_{\mathrm{L}}} d\tau = R\int_{f_{\mathrm{R}}}^{f_{\mathrm{L}}} df = R\int_{T_{\mathrm{R}}}^{T_{\mathrm{L}}} dT = (T_{\mathrm{L}} - T_{\mathrm{R}})R, \qquad (2)$$

where the subscripts "L" and "R" denote the values of the quantities on the left- and right-most segments in contact with the pulley.

It now becomes clear why the net torque $\tau_{\mathrm{net}} = (T_{\mathrm{L}} - T_{\mathrm{R}})R$ formula given in many introductory physics texts is correct despite the fact that it is the frictional force between the string and the pulley that actually exerts the torque on the pulley. The difference in tension between the ends of the string is due to the friction between the string and the pulley. When we calculate the total torque on the pulley due to this friction, the result is the same as the torque due to the difference of tensions between the left and right ends of the string.

One can perform a similar analysis to show that the tensions also do not exert a force on the pulley. However, when one calculates the actual forces on the pulley due to the normal and frictional forces, we again find that for a massless string the net force is the same as if the tensions T_{L} and T_{R} acted on the pulley.

So why do most textbooks fail to consider these issues when discussing pulley problems like this? One answer may be that this more complete analysis overly complicates the problem since both approaches give the same result. But as we have tried to point out, this can lead to conceptual issues within the mind of the student since the textbook analysis appears inconsistent with other statements made in the same book.

Is there a way to make the textbook treatment consistent while retaining its simplicity? One might consider redefining the system. If the string and pulley are considered as separate parts, then one must discuss the forces between them. However, Fig. 1 suggests a different approach. Suppose we instead consider as our system the pulley + segment of the string in contact with the pulley. Then if we consider the forces on this pulley-string segment system, the tensions T_{L} and T_{R} exerted by the left and right portions of the string that are not in contact with the pulley now are external forces, and the friction and normal forces no longer need to be considered since they are internal forces. Since the string is massless, the contact string segment does not modify the pulley's moment of inertia and so does not affect the pulley's angular acceleration. The analysis can then proceed as usually presented in the textbook, although one still needs to discuss why T_{L} and T_{R} are different.

To summarize, we hope that we have raised some issues when considering pulley problems involving massless strings like those found in most textbooks. While the general approach these books use leads to the correct answer, it may raise confusion in the minds of students since it appears inconsistent with concepts described elsewhere in the same book. A more careful treatment based on the well-known discussion from the capstan problem that we have reproduced here should be presented as an example so that students can understand why the simpler approach is valid. In addition, more care should be taken to emphasize the assumptions being made when arguing that the tension is constant throughout a massless string.

References

1. An Atwood's machine is a good example of such a problem; e.g., R. A. Serway and J. W. Jewett, Jr., *Principles of Physics*, 4th ed. (Thomson Brooks/Cole, Belmont, CA, 2006), pp. 310-311.
2. See, for example, R. D. Knight, *Physics for Scientists and Engineers: A Strategic Approach,* 2nd. ed. (Addison-Wesley, San Francisco, 2008), p. 196.
3. E.g., Ref. 2, p. 130 and p. 194.
4. G. L. Hazelton, "A force amplifier: The capstan," *Phys. Teach.* **14**, 432–433 (Oct. 1976).
5. R. A. Becker, *Introduction to Theoretical Mechanics* (McGraw-Hill, New York, 1954), pp. 45-47.
6. e.g., E. Levin, "Friction experiments with a capstan," *Am. J. Phys.* **59**, 80–84 (Jan. 1991).
7. W.-T. Lee and H.-W. Hsiao, "Tension of a soft spring in contact with a cylinder," *Phys. Teach.* **47**, 596–597 (Dec. 2009).
8. C. Bettis, "Capstan experiment," *Am. J. Phys.* **49**, 1080–1081 (Nov. 1981).

Dennis E. Krause *is an associate professor and chair of the physics department at Wabash College, where he has taught since 1998.*

Wabash College, Physics Department, Crawfordsville, IN 47933; kraused@wabash.edu

Yifei Sun *is a sophomore math and physics major at Wabash College from Zhengzhou, Henan, China. A question he raised in Krause's introductory mechanics course led to this paper.*